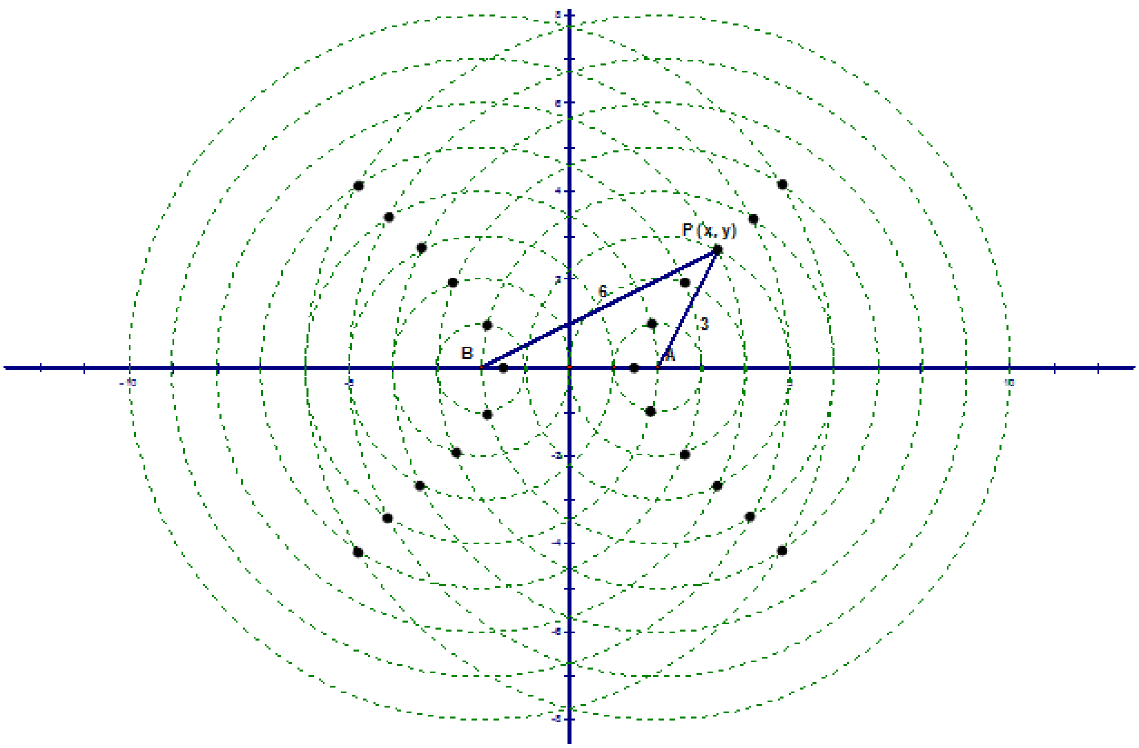
Hyperbolas

Standards Addressed
MGSE9-12.G.GPE.3 Derive the equations of ellipses and hyperbolas given the foci, using the fact that the sum or difference of distances from the foci is constant.

A hyperbola is the set of all points P in a plane where the absolute value of the differences of the distances from two fixed points (foci) remains constant. In the drawing shown, A and B are the fixed focus points and P is any point on a branch of the hyperbola where $|PA - PB| = 3$. Using the equally spaced circles centered at A and B, check the distances of each of the points marked from foci A and B, are the differences $|PA - PB|$ all equal to 3?

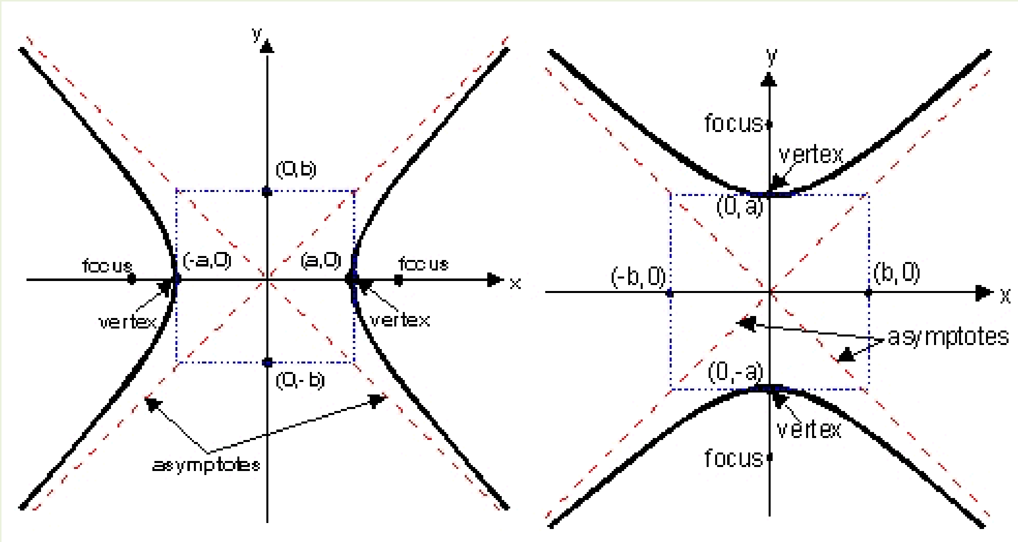


The midpoint of $\overline{AB}$ is the center of the hyperbola. For the hyperbola shown, the center is (0, 0). The vertices occur at the points of intersection of the branches and $\overline{AB}$. The segment joining the vertices is the transverse axis and the conjugate axis lies on the perpendicular bisector of the transverse axis.

Using the distance formula and the geometric definition of a hyperbola makes it possible to find to write an equation of the hyperbola in both general form and standard form.
Focus point A is located at (2, 0) and B is located at (-2, 0). The distance formula gives the following distances. As you work through the following procedures, describe what is being done in each step.

PA = $\sqrt{(x-2)^2+(y-0)^2}$ and PB = $\sqrt{(x+2)^2+(y-0)^2}$	apply the distance formula
then $\lvert PA-PB\rvert = 3$ and $PA-PB = \pm 3$	definition of absolute value
$\sqrt{(x-2)^2+(y-0)^2} - \sqrt{(x+2)^2+(y-0)^2} = \pm 3$	substitution
$\sqrt{(x-2)^2+(y-0)^2} = \pm 3 + \sqrt{(x+2)^2+(y-0)^2}$	separate the radical expressions
$\left(\sqrt{(x-2)^2+(y-0)^2}\right)^2 = \left(\pm 3 + \sqrt{(x+2)^2+(y-0)^2}\right)^2$	square both sides of the equation
$(x-2)^2 + y^2 = 9 \pm 6\sqrt{(x+2)^2+y^2} + (x+2)^2 + y^2$	begin to expand the right side
$x^2 - 4x + 4 + y^2 = 9 \pm 6\sqrt{(x+2)^2+y^2} + x^2 + 4x + 4 + y^2$	expand the binomials
$-8x - 9 = \pm 6\sqrt{(x+2)^2+y^2}$	simplify
$8x + 9 = \pm 6\sqrt{(x+2)^2+y^2}$	multiply both sides by -1
$(8x+9)^2 = \left(\pm 6\sqrt{(x+2)^2+y^2}\right)^2$	square again to remove the radical
$64x^2 + 144x + 81 = 36((x+2)^2+y^2)$	expand the binomials
$64x^2 + 144x + 81 = 36(x^2 + 4x + 4 + y^2)$	continue to expand the binomials
$64x^2 + 144x + 81 = 36x^2 + 144x + 144 + 36y$	multiply by 36
28x2 −36y2 = 63	combine like terms
$\frac{28}{63}x^2 - \frac{36}{63}y^2 = 1$	divide by 63
$\mathbf{\dfrac{x^2}{63/28} - \dfrac{y^2}{63/36} = 1}$	change to standard form of a hyperbola

Summary of Hyperbola Information



$Ax^2 - Cy^2 + Dx + Ey + F = 0$

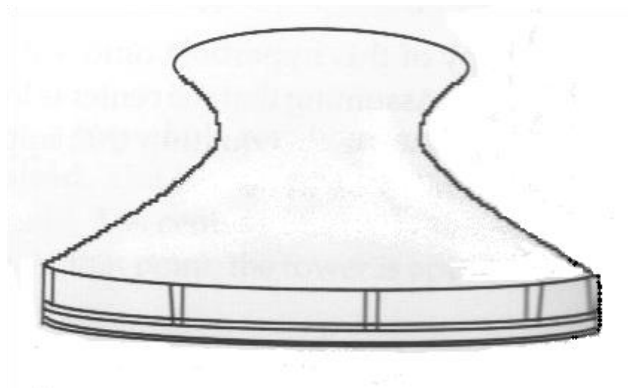
where $A \neq C$ and $AC < 0$

$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$	$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$
center: (h, k)	center: (h, k)
transverse axis: horizontal	transverse axis: vertical
length transverse axis: 2a	length transverse axis: 2a
length conjugate axis: 2b	length conjugate axis: 2b
asymptotes: $y = \pm \frac{b}{a} x$	asymptotes: $y = \pm \frac{a}{b} x$

$$c^2 = a^2 + b^2$$

vertices: $(h \pm a, k)$	vertices: $(h, k \pm a)$
co-vertices: $(h, k \pm b)$	co-vertices: $(h \pm b, k)$
foci: $(h \pm c, k)$	foci: $(h, k \pm c)$

Hyperbolic shapes are used for horns, street lamps, space heaters, and cooling towers for nuclear reactors. Rays emanating from one focus point, A , reflect off point P on the hyperbola as if they had emanated from the other focus point B. This has the effect of spreading out an waves coming from a focus point.

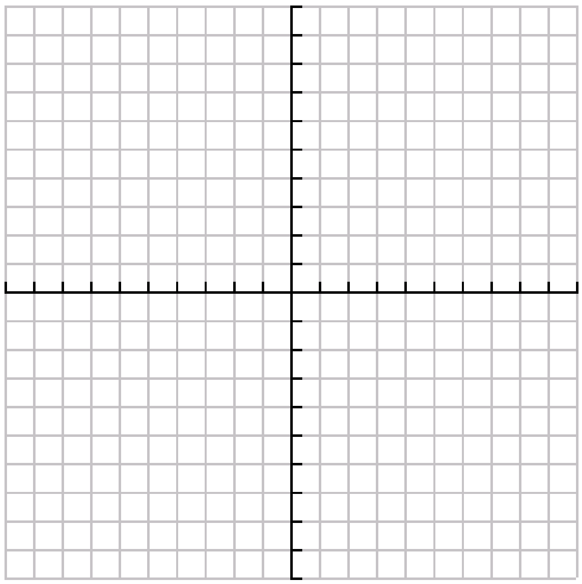


The architecture of the James S. McDonnell Planetarium of the St. Louis Science Center

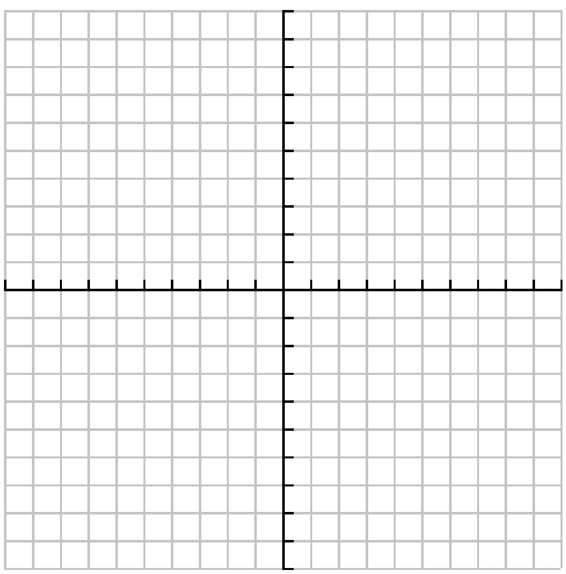
and the natural draft wet cooling hyperbolic towers at Didcot Power Station, UK show hyperbolic designs.

1. For each of the following hyperbolas, find the equation in standard form and draw the graph.

   a. vertices at (-3, 1) and (3, 1) and foci at (-5, 1) and (5, 1)

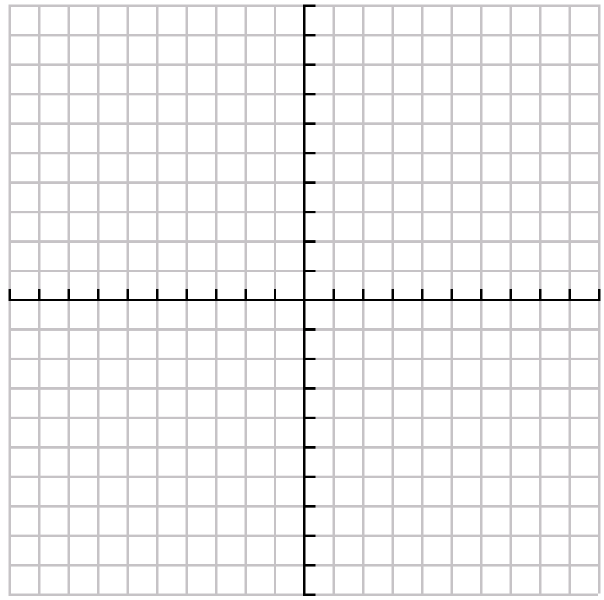


   b. vertices at (0, 1) and (0, -1) and foci at (0, $\sqrt{10}$) and (0, - $\sqrt{10}$)

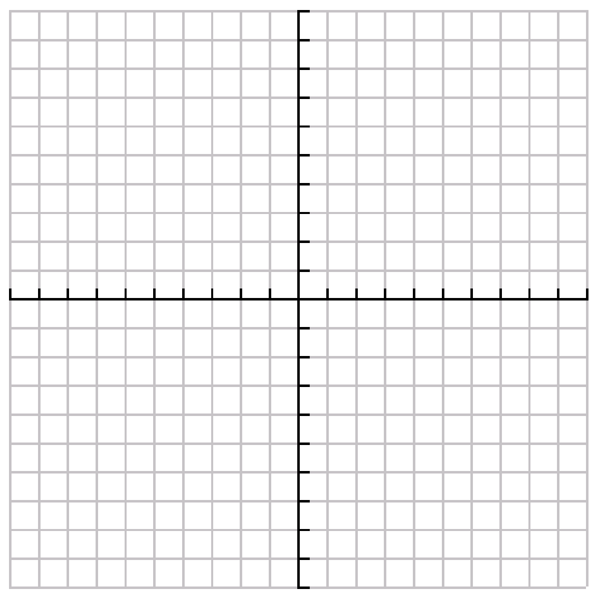


2. For each of the following hyperbolas, give the coordinates of the center, the foci, the vertices, and the co-vertices. Sketch the graph and include all of the values you found.

a. $4x^2 - 25y^2 = 100$



b. $\frac{(y-1)^2}{4} - \frac{(x+2)^2}{9} = 1$



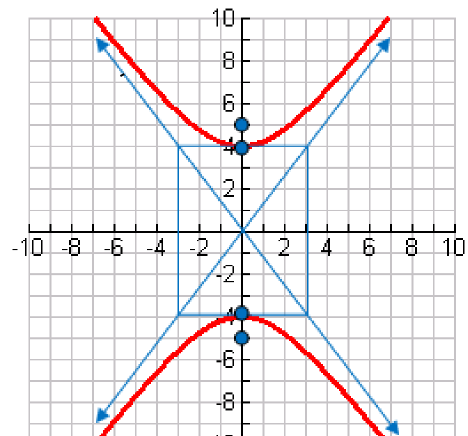
c. For the hyperbola graph, identify the foci, and vertices. Write an equation in standard form for the graph.

center

vertices

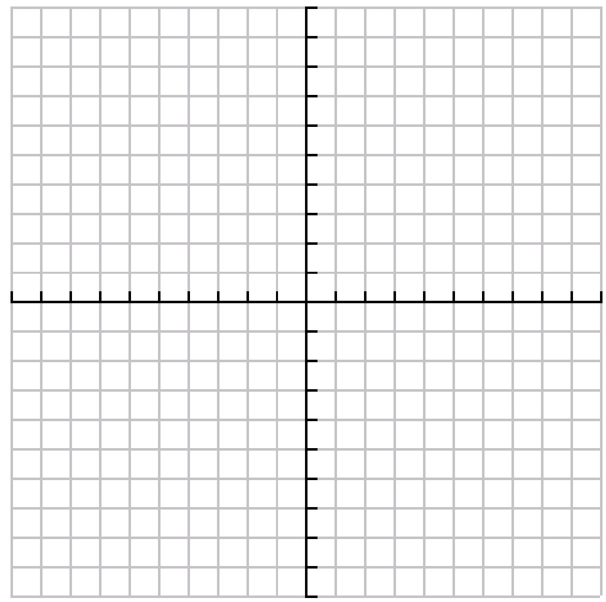
foci

equation

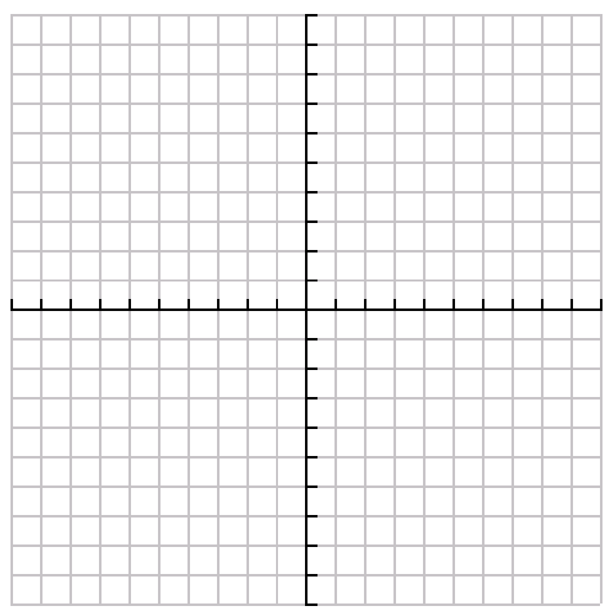


3. Write the standard equation for each hyperbola, give the coordinates of the center, vertices, co-vertices, and foci.

a. $4x^2 - 9y^2 - 8x + 54y - 113 = 0$



b. $y^2 - 9x^2 - 6y - 36x - 36 = 0$

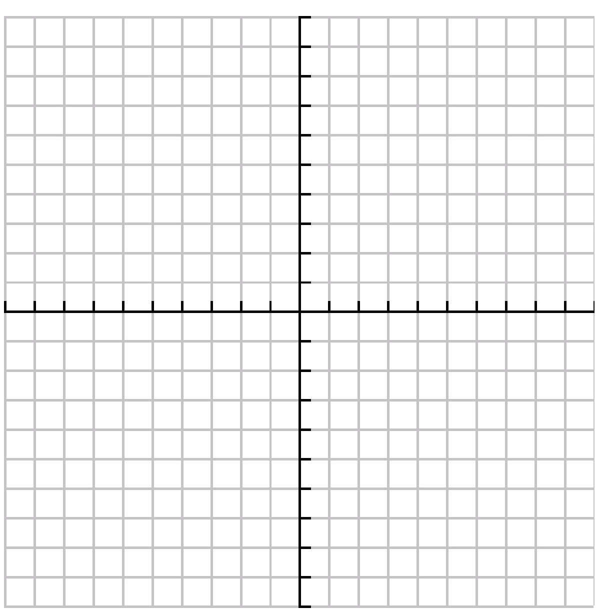


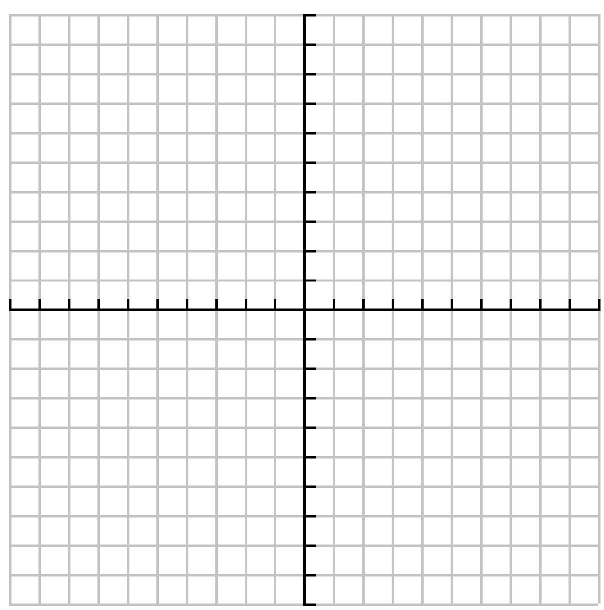
Hyperbola Practice

Identify the vertices, foci, and write the equation in standard form. Then sketch the graph.

1. $x^2 - y^2 - 4x - 6y - 9 = 0$

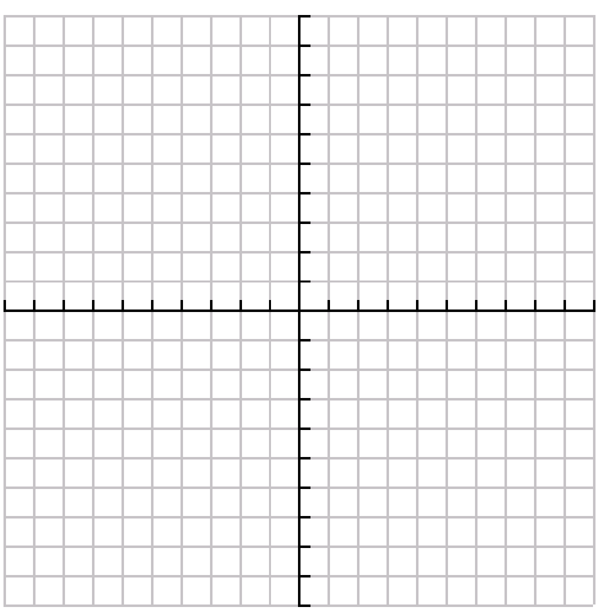
2. $-x^2 + y^2 - 9 = 0$

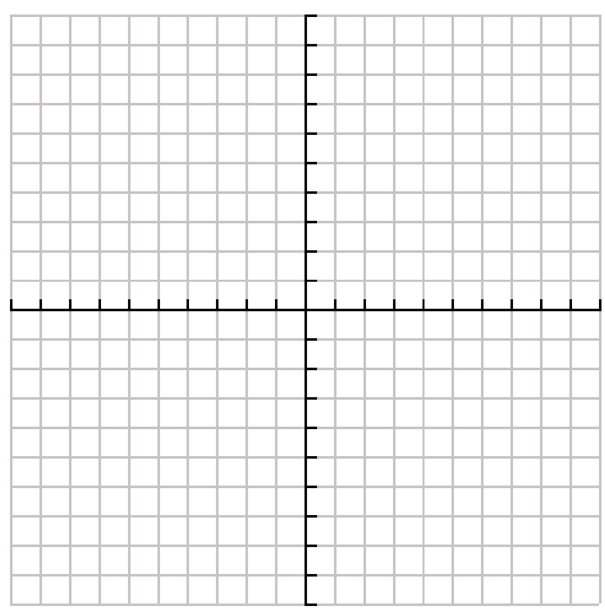




3. $-16x^2 + y^2 + 128x + 2y - 271 = 0$

4. $4x^2 - 9y^2 + 8x - 54y - 113 = 0$





5. Write the equation of the hyperbola with the following characteristics:

   Vertices (-5, -2) (1, -2) and Co-Vertices (-2, 4) (-2, -8)

6. Write the equation of the hyperbola with the following characteristics:

   Co-vertices (-6, 2) (-2, 2) and Foci (-4, $2 + \sqrt{29}$) (-4, $2 - \sqrt{29}$)