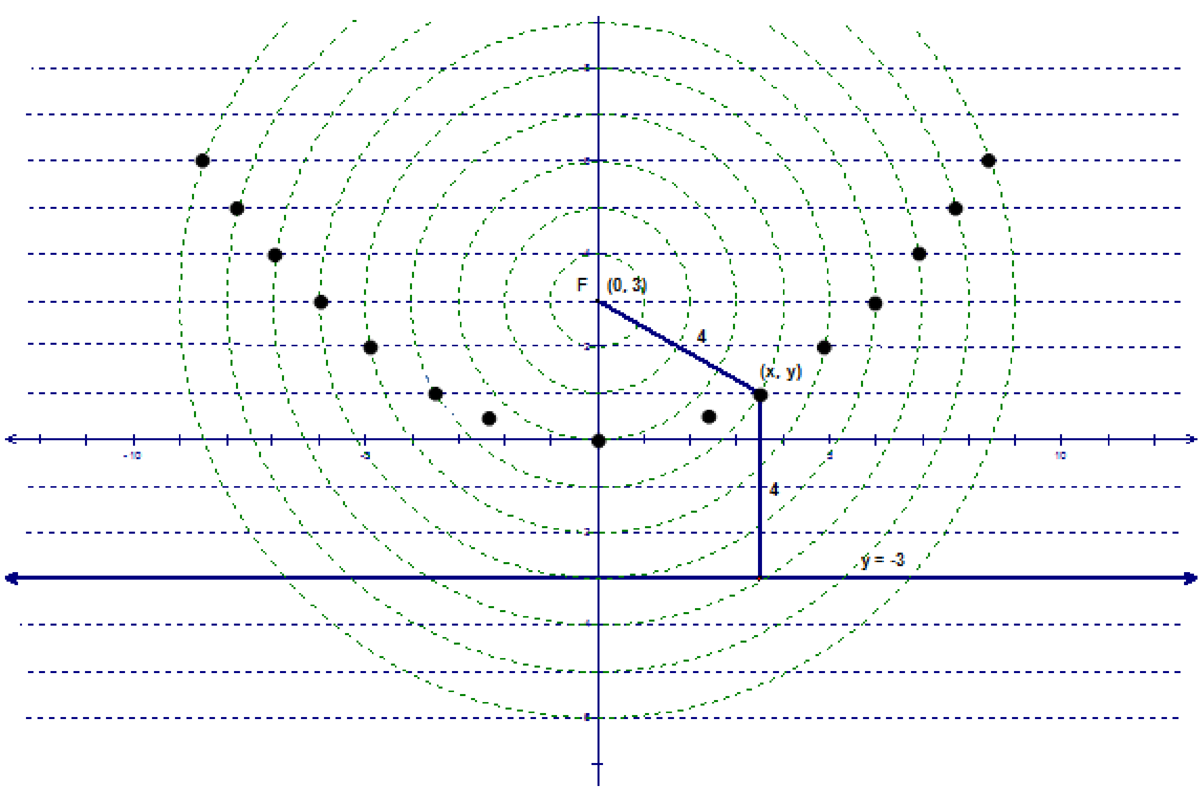
Parabolas

Standards Addressed
MGSE9-12.G.GPE.2 Derive the equation of a parabola given a focus and directrix.

Parabolas were studied in previous courses as quadratic functions where the equations were based on the position of the vertex and additional points were found using values of x on either side of the axis of symmetry. Equations were in the vertex form, ___________________, or in general quadratic form ______________________. We will use these forms and expand our study by including the geometric locus definition of the parabola.

The **locus** definition of the parabola is given in terms of a fixed point, the **focus**, and a fixed line, the **directrix**, on a plane. Each point on the parabola has a distance to the focus point which equals the distance from that point to the directrix. On the diagram below, point P is located at (x, y), the focus point is located at (0, 3), and the directrix has the equation y = -3. Note that any point on the directrix line can be written in the form (x, -3). The distances shown on the diagram are PF, from the parabola to the focus, and PD, from the parabola to the directrix, are both equal to 4. Similar distances from each point on the parabola have the same relationship, PF = PD, but are different lengths.

Draw segments from several of the points on the parabola to the focus and to the directrix to convince yourself that the relationship PF = PD holds for all points.



Using the relationship PF = PD, distance formula, the points P(x, y), F(0, 3), and D(x, -3), we can find an algebraic equation for the parabola. As you work through the following procedures, describe what is being done in each step.

PF = $\sqrt{(x-0)^2+(y-3)^2}$ and PD = $\sqrt{(x-x)^2+(y+3)^2}$ apply the distance formula

$\sqrt{(x-0)^2+(y-3)^2} = \sqrt{(x-x)^2+(y+3)^2}$ substitution

$\sqrt{x^2+(y-3)^2} = \sqrt{(y+3)^2}$ simplify

$x^2 + (y - 3)^2 = (y + 3)^2$ square both sides of the equation

$x^2 = (y + 3)^2 - (y - 3)^2$ subtract (y − 3)2 from both sides

$x^2 = y^2 + 6y + 9 - (y^2 - 6y + 9)$ expand the binomial terms

$x^2 = 12y$ simplify

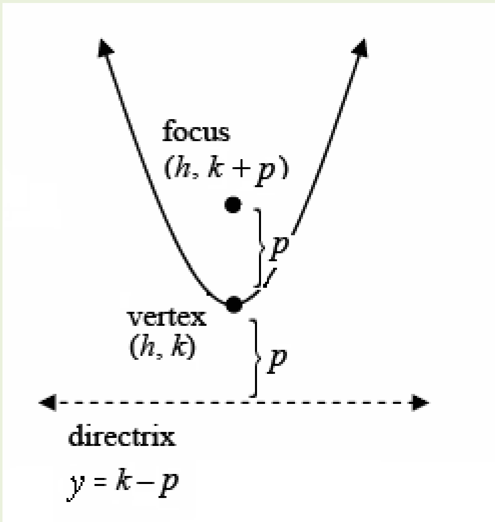
$y = \frac{1}{12}x^2$ division

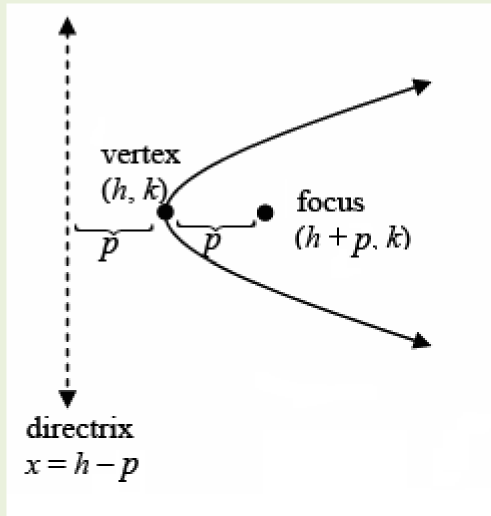
$y=\frac{1}{12}x^2$ can be written as $y = \frac{1}{12}(x-0)^2+0$ the vertex form, where the vertex is (0, 0) and

$a=\frac{1}{12}$, a positive value, meaning the parabola opens upward. $y=\frac{1}{12}x^2$ also fits the general form with $a=\frac{1}{12}$, b = 0 and c = 0. These observations all agree with the parabola shown in the starting diagram.

Adding the focus and the directrix gives more information about the parabola. The distance from the focus point at (0, 3) to the vertex at (0, 0) is 3 units and the distance from the vertex to the directrix is also 3 units. These distances are generalized to give the **position of the focus as (0, p)**, the **equation of the directrix as y = -p**, and the equation of the parabola as **y =** $\frac{1}{p}x^2$**.** The parabola equation **y =** $\frac{1}{12}x^2$ is y = $\frac{1}{4(3)}x^2$**.** Parabolas can open horizontally as well as vertically and the information on parabolas with vertices at (0, 0) are summarized in the following table.

Summary of Parabola Information





$Ax^2 + Cy^2 + Dx + Ey + F = 0$

where A = 0 or C = 0 but not both = 0

vertex at (h, k)

Opens Vertically

$$4p(y-k)=(x-h)^2$$

$p > 0$ opens up
$p < 0$ opens down

focus (h, k + p)

directrix y = k − p

axis of symmetry x = h

Opens Horizontally

$$4p(x-h)=(y-k)^2$$

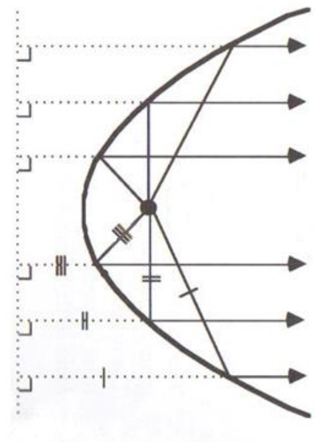
$p > 0$ opens right
$p < 0$ opens left

focus (h + p, k)

directrix x = h − p

axis of symmetry y = k

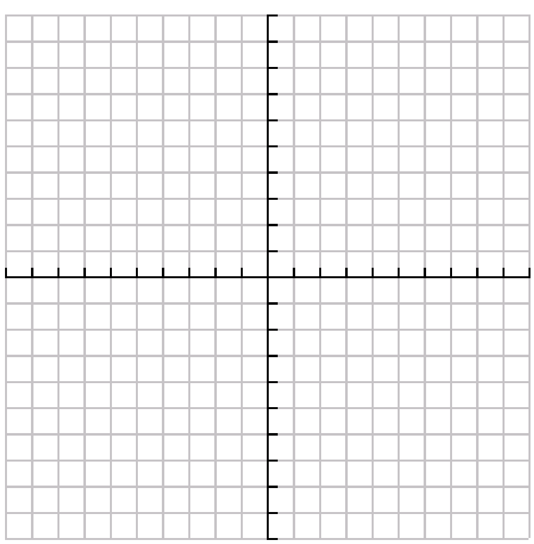
Parabolas have special reflective properties which make them useful shapes for many items including flashlights, car headlights, suspension bridges such as the Golden Gate Bridge, solar cookers, and satellite dishes. Two properties are especially important when considering applications of parabolas. First, all rays in the interior of a parabola parallel to the axis of symmetry are reflected toward the focus. And, all rays emitted from the focus are reflected so that each reflected ray runs parallel to the axis of symmetry and perpendicular to the directrix.

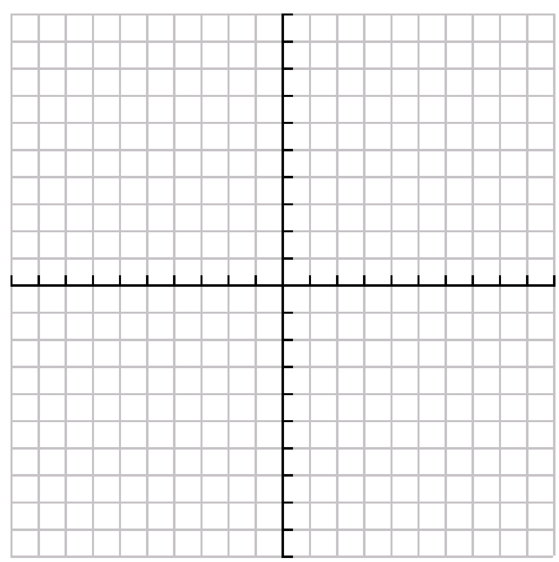


1. In each of the following problems, list the coordinates of the vertex, the coordinates of the focus, the equation of the directrix, and graph the parabola.

a. $(y - 3)^2 = -12(x + 2)$

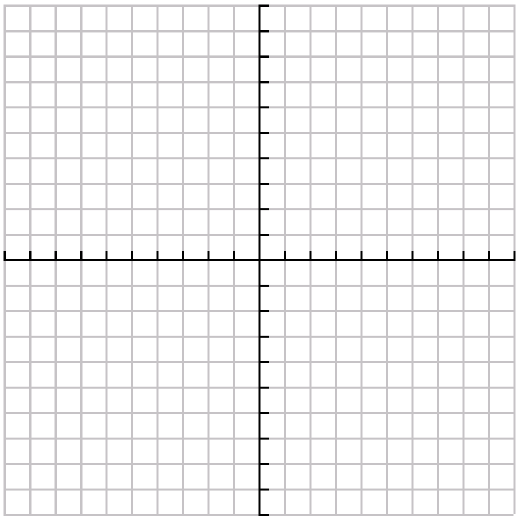
b. $(x - 1)^2 = -8y$

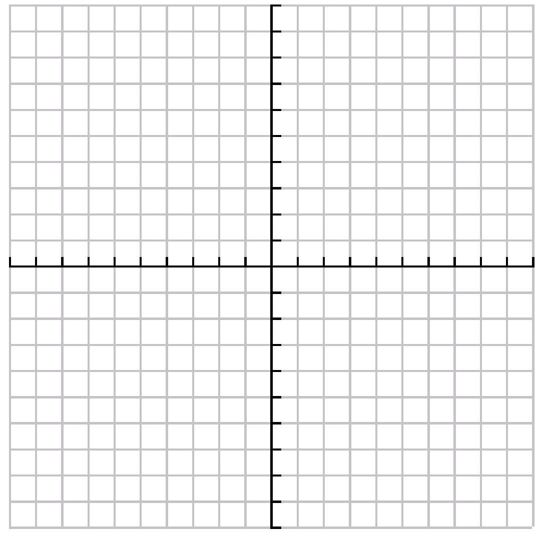




c. $x = (y - 4)^2$

d. $(x + 1)^2 = 2(y + 3)$



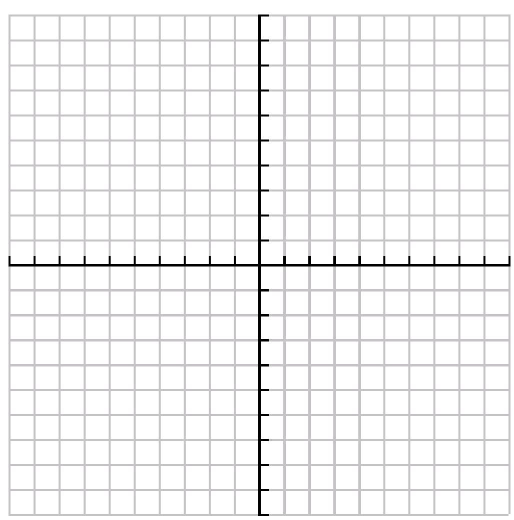


To change parabolas given in general form to standard form to facilitate graphing, it is necessary to complete the square. Given the parabola $2x^2 - 4x + y + 4 = 0$, first rewrite the equation with x terms and y terms on different sides of the equation.

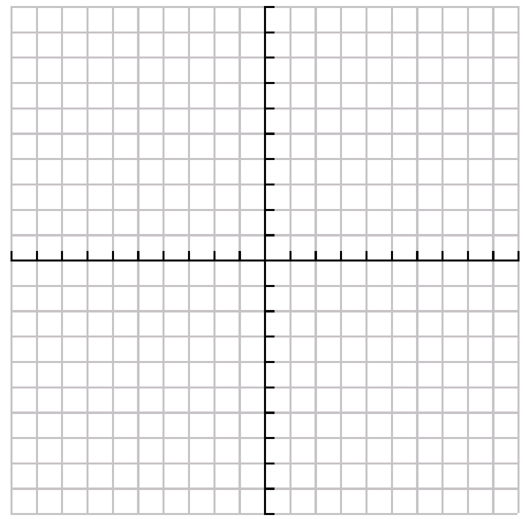
$2x^2 - 4x + y + 4 = 0$	
$y + 4 = -2x^2 + 4x$	separate x terms and y terms
$y + 4 = -2(x - 2x)$	prepare to complete the square by factoring out -2
$y + 4 - 2 = -2(x^2 - 2x + 1)$	complete the square; add -2 to both sides
$y + 2 = -2(x - 1)^2$	factor

2. Write each of the following equations in standard form. List the vertex, coordinates of the focus and equation of the directrix. Graph the parabola.

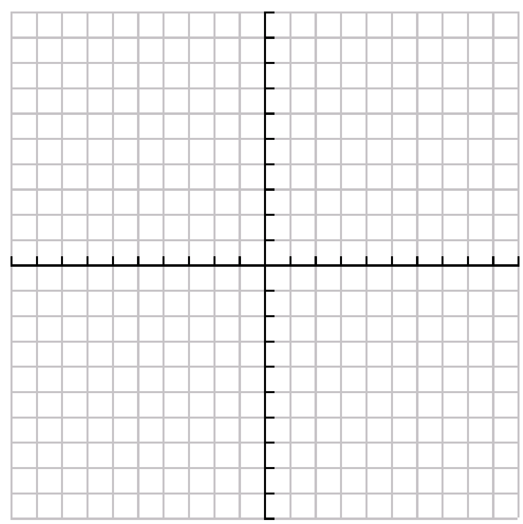
a. $y^2 - 8y + 8x + 8 = 0$



b. $x^2 - 6x + 12y + 21 = 0$



c. $2y^2 - 8y - 4x + 12 = 0$



3. Write the equation of a parabola with a vertex of (5, 2) and a directrix of $y = -2$.

4. Write the equation of a parabola with a vertex of (-3, 4) and a directrix of $x = -6$

5. Write the equation of a parabola with a vertex of (5, -1) and a focus of (0, -1).

6. Write the equation of a parabola with a vertex of (-1, -2) and a focus of (-1, -4).

7. Write the equation of a parabola with a focus of (0, 2) and a directrix of $x = 8$.

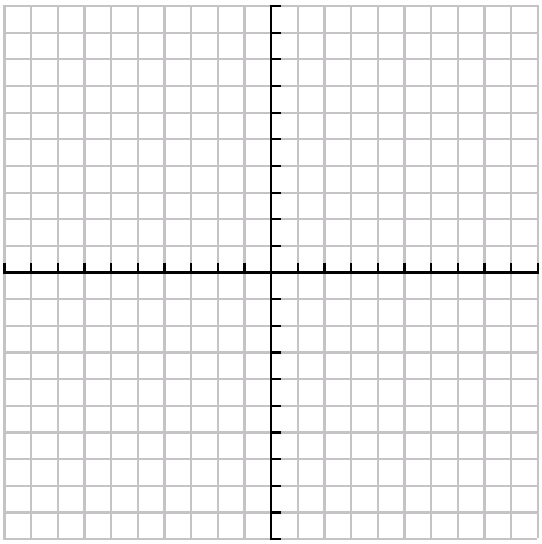
8. Write the equation of a parabola with a focus of (4, -1) and a directrix of $y = 5$.

Parabola Practice

Write the equation of the parabola in standard form. Identify the directrix and focus and sketch the graph.

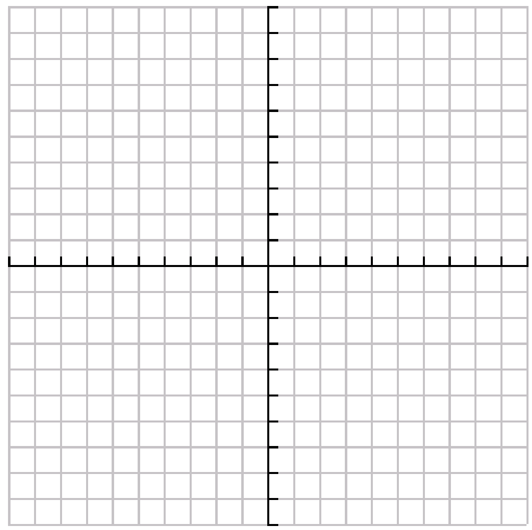
1. $y^2 - 4y + 12x - 8 = 0$

Vertex:__________
Directrix: ________
Focus: __________



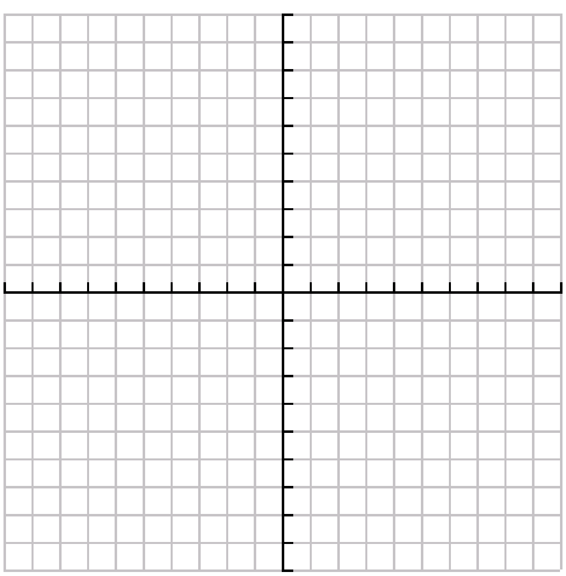
2. $x^2 + 4x - 8y + 12 = 0$

Vertex:___________
Directrix: _________
Focus: ___________



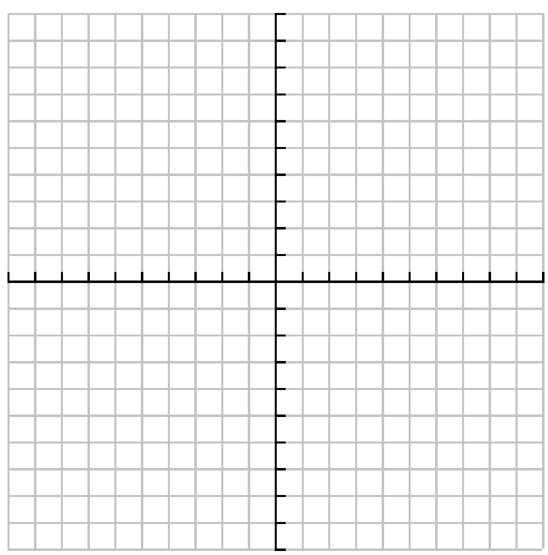
3. $2y^2 - 8y - 14x - 20 = 0$

Vertex:__________
Directrix: ________
Focus: __________

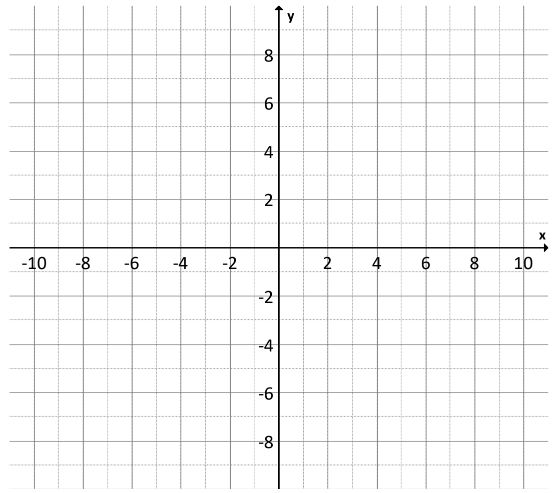


4. $x^2 + 6x - 2y + 13 = 0$

Vertex:___________
Directrix: _________
Focus: ___________



5. Graph a parabola and determine its equation with a vertex at $(1,5)$ and a focus at $(1,7)$



6. Graph a parabola and determine its equation with a vertex at $(-3,1)$ and directrix at x = -8.